Invariants of the LWIR Thermophysical Model

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Abstract

The temperature of a surface viewed by a long-wave infrared camera can be predicted by a thermophysical model (a conservation of energy statement at the surface of a unit volume). However, this prediction currently requires at least 24 hours of previous imagery in order to estimate the parameters of the model. Absolute invariants, relative invariants, and quasi-invariants provide three possible methods for circumventing this obstacle. Lie group analysis is a fundamental tool for systematically exploring invariance and for finding the appropriate transformations groups. This paper discusses the relevant parts of Lie group analysis and uses them to find the transformation groups and absolute invariants of the thermophysical model. The goal is to recognize objects based upon a composition of materials that are identified using invariant features of infrared imagery.

1. Introduction

Lie group analysis provides a constructive method for determining the *transformation groups and invariants* defined on a manifold M_f determined by a model equation. Although the theory discussed will be applied to LWIR sensors, this technique is not sensor specific, therefore it is applicable to any sensor available (MMW, EO, IR, RADAR, etc.). Invariant features provide the foundation upon which object recognition systems may be rigorously and soundly constructed. Given an invariant determined analytically, the constant value an invariant function assumes for a given class can be determined by a single measurement. In practice, to account for the variability in the measurement process, several measurements would be taken to determine a sample mean for the constant value.

This paper analyzes the (thermophysical) conservation of energy model with Lie groups. From this model, it is possible to derive the transformation groups and invariants using Lie group analysis. Therefore, the independence and linear transformation assumptions made in previous works will no longer be required [5].

2. The Thermophysical Model

The thermophysical model (Figure 1) is based on the conservation of energy law from heat transfer theory

$$\begin{aligned} f &\equiv W_{\text{absorb}} - W_{\text{radiate}} - W_{\text{convect}} - W_{\text{conduct}} - W_{\text{store}} \\ &\equiv [W_s \alpha_s \cos\theta + W_l \alpha_l A_{\text{sky}}] - [\epsilon\sigma T_s^4] \\ &\qquad -[-h(T_\infty - T_s)] - [-k\frac{\partial T_s}{\partial z}] - [C_T \frac{\partial T_s}{\partial t}] \\ &= 0. \end{aligned} \tag{1}$$

Each component is a heat flux ($\frac{W}{m^2}$) and the variables are further described in [2] and [1].

3. Lie Group Analysis

The following discourse is a summary of the fundamental components of Lie group analysis. Theorem 16 shows an invariant function is induced by an associated group action. This theorem is usually taken as a definition [6, 7]. An exception is Diudonné [4] who used the concept of induced group actions to determine invariants. This is a self-contained discourse that attempts to touch only briefly those components of Lie group analysis that are applicable to algebraic equations of the form

$$f(\overline{x}) = 0 \tag{2}$$

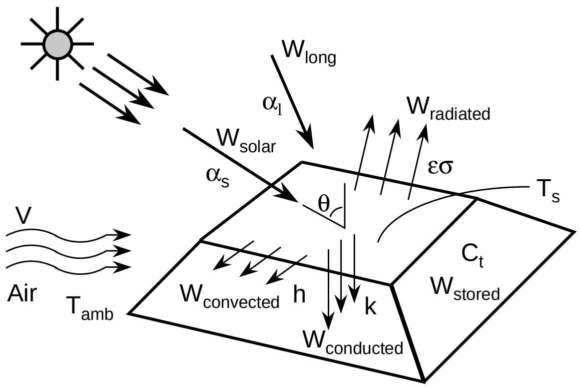

Figure 1. Energy exchange at the surface of a viewed object. Incident energy is primarily in the visible spectrum. Surfaces lose energy by convection to air, via radiation to the atmosphere, and via conduction to the interior of the object. A storage term accounts for any residual energy.

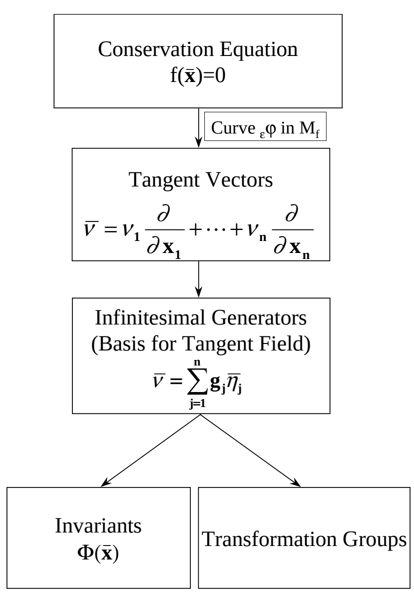

Figure 2. Top down view of the major steps involved in Lie group analysis of algebraic (non-differential) equations. Each step (and supporting math notation) is discussed in section 3.

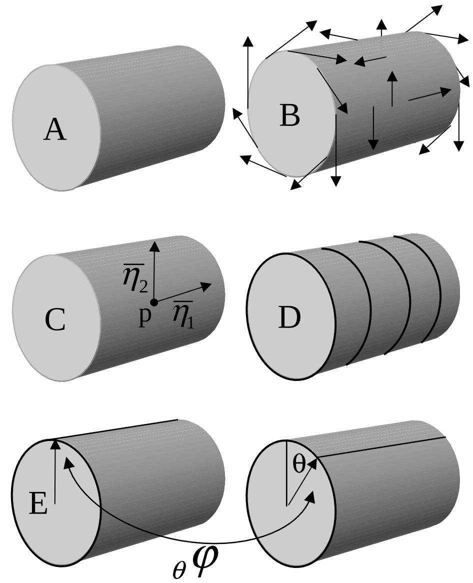

Figure 3. An intuitive illustration of Lie group analysis. (A) the manifold defined by a conservation equation (ex. surface of a cylinder). (B) Tangent vectors on the manifold. (C) A Basis for the Tangent Space at point p. (D) Slices of the cylinder (circles) represent Absolute Invariants, Φ, under rotation. (E) Rotations by θ represent a transformation group.

where $\overline{x} = (x_1, \ldots, x_n) \in \Re^n$ and f is a differentiable function, $f \in C^1(\Re)$. Denote the set of roots of f by

$$M_f \equiv \{\overline{x} \in \Re^n : f(\overline{x}) = 0\}. \tag{3}$$

If the differential $df \neq 0 \quad \forall \overline{x} \in M_f$ then f implicitly defines an $n-1$ dimensional manifold. We assume this manifold to be connected[1].

A symmetry of M_f is a bijective mapping g of M_f such that if $f(\overline{x}) = 0$ then $f(g\overline{x}) = 0$. Lie group analysis will determine continuous symmetries only; if the manifold is not connected, discrete symmetries may exist and cannot be determined by the methods considered here. An example of a discrete symmetry is reflection. For physical systems, such as object recognition, discrete symmetries are generally not an issue.

In general, Lie group analysis is applicable for *systems of equations*, however, any system of equations $g_i = 0$ for $i = 1, \ldots, m$ can be replaced by a single equation $f \equiv \sum_{i=1}^m g_i^2 = 0$ in the sense that $M_{g_1,\ldots,g_m} = M_f$. Hence, there is no loss of generality in assuming only one equation. Figure 3 graphically illustrates Lie group analysis.

[1] A manifold, M, is connected if to each pair of points in M there exists a curve in M connecting the two points.

3.1. Curves and Groups of Transformations

A *curve* in $\Re^n$ is a differentiable function

$$\begin{aligned} {}_\bullet\varphi &: I \to \Re^n \\ &: \varepsilon \mapsto (\alpha_1, \ldots, \alpha_n) \end{aligned}$$

where $I \subseteq \Re$ is an open interval and $\alpha_j \in \Re$ for $j = 1, \ldots, n$. A curve in M_f is a curve in $\Re^n$ whose image lies in M_f. By the definition of M_f this implies ${}_\varepsilon\varphi$ is a curve in M_f if and only if

$$f({}_\varepsilon\varphi) = 0 \qquad \forall \varepsilon \in I.$$

A curve ${}_\varepsilon\varphi$ in $\Re^n$ determines n-component functions ${}_\varepsilon\varphi_{x_j}$ by

$${}_\varepsilon\varphi_{x_j} \equiv \pi_j \circ {}_\varepsilon\varphi$$

where

$$\begin{aligned} \pi_j(\bullet) &: \Re^n \to \Re \\ &: (\alpha_1, \ldots, \alpha_n) \mapsto \alpha_j \end{aligned}$$

is the j^{th} projection function. Since ${}_\varepsilon\varphi$ is differentiable each component function ${}_\varepsilon\varphi_{x_j}$ must necessarily be differentiable, so ${}_\varepsilon\varphi_{x_j} \in C^1(I)$. Thus a curve ${}_\varepsilon\varphi$ can be written as

$$\begin{aligned} {}_\bullet\varphi &: I \to \Re^n \\ &: \varepsilon \mapsto ({}_\varepsilon\varphi_{x_1}, \ldots, {}_\varepsilon\varphi_{x_n}) \end{aligned}$$

where each ${}_\varepsilon\varphi_{x_j} \in C^1(I)$.

If $({}_\varepsilon\varphi_{x_1}, \ldots, {}_\varepsilon\varphi_{x_n})$ is a vector field on $\Re^n$ (so ${}_\varepsilon\varphi_{x_j} = {}_\varepsilon\varphi_{x_j}(\overline{x}), \overline{x} \in \Re^n$) then for each fixed ε

$${}_\varepsilon\varphi(\overline{x}) \equiv ({}_\varepsilon\varphi_{x_1}(\overline{x}), \ldots, {}_\varepsilon\varphi_{x_n}(\overline{x})) \in \underbrace{C^1(\Re^n) \times \ldots \times C^1(\Re^n)}_{n \text{ factors}},$$

so each ${}_\varepsilon\varphi(\overline{x})$ determines a transformation map of $\Re^n$ given by

$$\begin{aligned} {}_\varepsilon\varphi(\bullet) &: \Re^n \to \Re^n \\ &: \overline{x} \mapsto ({}_\varepsilon\varphi_{x_1}(\overline{x}), \ldots, {}_\varepsilon\varphi_{x_n}(\overline{x})). \end{aligned}$$

As ε varies over I this determines a family of transformations $\{{}_\varepsilon\varphi(\overline{x})\}_{\varepsilon \in I}$.

If the *evaluation function* at $\overline{x}$ is defined as

$$\begin{aligned} e_{\overline{x}}(\bullet) &: \underbrace{C^1(\Re^n) \times \ldots \times C^1(\Re^n)}_{n \text{ factors}} \to \Re^n \\ &: (f_1, \ldots, f_n) \mapsto (f_1(\overline{x}), \ldots, f_n(\overline{x})) \end{aligned}$$

then for a fixed ε,

$${}_\varepsilon\varphi(\overline{x}) \equiv e_{\overline{x}}({}_\varepsilon\varphi) = ({}_\varepsilon\varphi_{x_1}(\overline{x}), \ldots, {}_\varepsilon\varphi_{x_n}(\overline{x})) \in \Re^n.$$

If $\overline{x}$ is treated as a fixed constant then a curve is determined as ε varies over I by

$$\begin{aligned} {}_\bullet\varphi(\overline{x}) \equiv e_{\overline{x}}({}_\bullet\varphi) &: I \to \Re^n \\ &: \varepsilon \mapsto ({}_\varepsilon\varphi_{x_1}(\overline{x}), \ldots, {}_\varepsilon\varphi_{x_n}(\overline{x})). \end{aligned}$$

As $\overline{x}$ varies over $\Re^n$, ${}_\varepsilon\varphi(\overline{x})$ determines a family of curves, $\{{}_\varepsilon\varphi(\overline{x})\}_{\overline{x} \in \Re^n}$, one for each point $\overline{x} \in \Re^n$.

Example 1

Consider $f(x, y, z) = x^2 + y^3 - z + 1$. The curve

$$\begin{aligned} {}_\bullet\varphi(\overline{x}) &: \Re \to \Re^3 \\ &: \varepsilon \mapsto (x + \varepsilon, y, \varepsilon^2 + 2x\varepsilon + z) \end{aligned}$$

is a curve in M_f for any $\overline{x} = (x, y, z) \in M_f$ since

$$\begin{aligned} f({}_\varepsilon\varphi(\overline{x})) &= ({}_\varepsilon\varphi_x(\overline{x}))^2 + ({}_\varepsilon\varphi_y(\overline{x}))^3 - ({}_\varepsilon\varphi_z(\overline{x})) + 1 \\ &= (x + \varepsilon)^2 + y^3 - (\varepsilon^2 + 2x\varepsilon + z) + 1 \\ &= x^2 + y^3 - z + 1 \\ &= 0. \end{aligned}$$

If one fixes $\varepsilon = 1$ then

$$\begin{aligned} {}_1\varphi(\bullet) &: M_f \to M_f \\ &: (x, y, z) \mapsto (x + 1, y, 1 + 2x + z) \end{aligned}$$

determines a transformation of M_f.

The set of transformations $\{{}_\varepsilon\varphi(\overline{x})\}_{\varepsilon \in I}$ has a natural binary operation defined on it given by composition

$$\begin{aligned} ({}_\varepsilon\varphi \circ {}_\delta\varphi)(\bullet) &: \Re^n \to \Re^n \\ &: \overline{x} \mapsto {}_\varepsilon\varphi({}_\delta\varphi(\overline{x})). \end{aligned}$$

A *group of transformations* $\{{}_\varepsilon\varphi(\overline{x})\}_{\varepsilon \in I}$ is a set of transformations such that the operation of composition satisfies

i. associativity, ${}_\varepsilon\varphi \circ ({}_\delta\varphi \circ {}_\gamma\varphi) = ({}_\varepsilon\varphi \circ {}_\delta\varphi) \circ {}_\gamma\varphi$
ii. there exist an identity element ${}_0\varphi$, and
iii. each element in $\{{}_\varepsilon\varphi(\overline{x})\}_{\varepsilon \in I}$ has an inverse.

The transformation ${}_\varepsilon\varphi(\overline{x})$ is a parameterized transformation of $\Re^n$. Since it has a single parameter, the group of transformations $\{{}_\varepsilon\varphi(\overline{x})\}_{\varepsilon \in I}$ is called a one-parameter group of transformations.

A one-parameter group of transformations $\{{}_\varepsilon\varphi(\overline{x})\}_{\varepsilon \in I}$ is often called a *flow* because this "visually depicts" these transformations: Consider a particle at point $\overline{x}$ in $\Re^n$ at time $\varepsilon = 0$. As ε varies the curve ${}_\varepsilon\varphi(\overline{x})$ traces out a trajectory in $\Re^n$. This trajectory is the *flow* of the particle under the curve ${}_\varepsilon\varphi$.

A one-parameter *Lie Group* is a group that also carries the structure of a 1-dimensional differentiable manifold. This additional structure on a group allows the ability to speak of continuity and differentiability.

3.2. Tangent Vectors and Vector Fields

A *tangent vector* consists of a vector part and a point of application. We denote a tangent vector by $\overline{v}_{\overline{x}} = (v_1, v_2, \ldots, v_n)_{\overline{x}}$ where $(v_1, v_2, \ldots, v_n)$ is "the vector part" and $\overline{x}$ is the point of application. If ${}_\varepsilon\varphi$ is a curve then $\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=a} {}_\varepsilon\varphi$ determines a tangent vector at ${}_a\varphi$.

Each tangent vector $\overline{v}_{\overline{x}}$ determines a map by

$$\begin{aligned} \phi_{\overline{v}_{\overline{x}}}(\bullet) &: C^1(\Re^n) \to \Re \\ &: f \mapsto \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} f(\overline{x} + \varepsilon\overline{v}_{\overline{x}}) \end{aligned}$$

where

$C^1(\Re^n) \equiv$ The set of differentiable functions on $\Re^n$.

For brevity, one simply writes

$$\overline{v}_{\overline{x}}(f) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} f(\overline{x} + \varepsilon\overline{v}_{\overline{x}}).$$

Example 2

If $f(x,y,z) = x^2 + xy + yz + z^2 - 1$, $\overline{v}_{\overline{x}} = (1,2,0)$ and $\overline{x} = (x,y,z) = (1,0,1)$ then

$$\overline{x} + \varepsilon\overline{v}_{\overline{x}} = (1+\varepsilon, 2\varepsilon, 1)$$

so

$$\begin{aligned}\overline{v}_{\overline{x}}(f) =& \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \{(1+\varepsilon)^2 + (1+\varepsilon)(2\varepsilon) + (2\varepsilon)(1) + (1)^2 - 1\}\\ =& \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \{3\varepsilon^2 + 6\varepsilon + 2\}\\ =& \{6\varepsilon + 6\}_{\varepsilon=0}\\ =& 6.\end{aligned}$$

It is an easy exercise to show that $\overline{v}_{\overline{x}}(f) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} f(\overline{x} + \varepsilon\overline{v}_{\overline{x}}) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} f({}_\varepsilon\varphi)$ for any curve ${}_\varepsilon\varphi$ through the point $\overline{x}$ satisfying $\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} {}_\varepsilon\varphi_{x_j} = v_j$.

The set of all tangent vectors at a point $\overline{x} \in \Re^n$ forms a vector space of dimension n, called the tangent space of $\Re^n$ at $\overline{x}$, which is denoted $T_{\overline{x}}(\Re^n)$. If ${}_0\varphi(\overline{x}) = \overline{x}$ then one says "${}_\varepsilon\varphi(\overline{x})$ is a curve at $\overline{x}$". Let ${}_\varepsilon\varphi(\overline{x})$ be a curve at $\overline{x}$. Then $\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} {}_\varepsilon\varphi(\overline{x}) \in T_{\overline{x}}(\Re^n)$. It takes n values to specify the components of the tangent vector uniquely at $\overline{x}$. A vector field on M_f is a function ${}_\varepsilon\varphi$ that assigns to each point $x \in M_f \subseteq \Re^n$ a vector in $\Re^n$

$${}_\varepsilon\varphi : M_f \to \Re^n.$$

A set of tangent vectors $\{\overline{v}_{\overline{x}}\}_{\overline{x}\in\Re^n}$ on $\Re^n$ determines a vector field $\overline{v}$ on $\Re^n$

$$\begin{aligned}\overline{v}_\bullet \quad &: \quad \Re^n \to \dot{\bigcup}_{\overline{x}\in\Re^n} T_{\overline{x}}(\Re^n)\\ &: \quad \overline{x} \mapsto \overline{v}_{\overline{x}}\end{aligned}$$

where the disjoint union

$$\dot{\bigcup}_{\overline{x}\in\Re^n} T_{\overline{x}}(\Re^n)$$

is called the *tangent bundle* of $\Re^n$. It forms a manifold of dimension $2n$. The tangent vector $\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} {}_\varepsilon\varphi(\overline{x}) \in T_{\overline{x}}(\Re^n) \subseteq \dot{\bigcup}_{\overline{x}\in\Re^n} T_{\overline{x}}(\Re^n)$ and requires $2n$ values to uniquely specify it, n values to specify the components of the tangent vector at $\overline{x}$, and n values to specify $\overline{x}$ itself.

A vector field $\overline{v} = (v_1, \ldots, v_n)$ on $\Re^n$ is determined by n component functions $v_j \in C^1(\Re^n)$. The relation between a vector field and a tangent vector at $\overline{x}$ determined by the vector field is just evaluation

$$e_{\overline{x}}(\overline{v}) = \overline{v}_{\overline{x}}.$$

Example 3

Given the vector field $\overline{v} = (x^2, 3x - z, y)$ and point $\overline{x} = (1,0,1)$, the evaluation function determines the tangent vector

$$e_{\overline{x}}(\overline{v}) = \overline{v}_{\overline{x}} = (1,2,0).$$

For $f(x,y,z) = x^2 + xy + yz + z^2 - 1$,

$$\overline{v}_{\overline{x}}(f) = 6$$

the same result as previously calculated.

Lemma 4 *Let $\overline{v} = (v_1, v_2, \ldots, v_n)$ be a vector field and $f \in C^1(\Re^n)$. Then*

$$\overline{v}(f) = \sum_{j=1}^n v_j \frac{\partial f}{\partial x_j}. \tag{4}$$

Proof: Apply the chain rule. ∎

By this lemma, it is meaningful to write

$$\overline{v}(f) = \left(\sum_{j=1}^n v_j \frac{\partial}{\partial x_j}\right) f$$

and make the identification

$$\overline{v} = (v_1, v_2, \ldots, v_n) \leftrightarrow \sum_{j=1}^n v_j \frac{\partial}{\partial x_j}.$$

This equivalence between vector fields and tangent vectors can be formalized using the concept of derivations [3].

Thus a tangent vector, and therefore vector fields as well, can be viewed as either an ordered *n–tuple* or as an operator. It is this ability to view tangent vectors (vector fields) from both perspectives that makes them so powerful.

3.3. Tangent Fields and Infinitesimal Generators

The set of vector fields over $\Re^n$ consisting of elements

$$\overline{v} = (v_1, v_2, \ldots, v_n)$$

where

$$v_j = v_j(\overline{x}) \in C^1(\Re^n)$$

form a module over the ring $C^1(\Re^n)$ with scalar multiplication being componentwise. Since

$$\begin{aligned}(g\overline{v})(\bullet) \quad &: \quad C^1(\Re^n) \to C^1(\Re^n)\\ &: \quad f \mapsto (g\overline{v})(f)\end{aligned}$$

where

$$\begin{aligned}[(g\overline{v})f](\bullet) \quad &: \quad \Re^n \to \Re\\ &: \quad \overline{x} \mapsto (g(\overline{x})\overline{v}_{\overline{x}})f\end{aligned}$$

the set of all vector fields satisfying

$$\overline{v}(f) = 0$$

form a submodule since

$$\overline{v}(f) = 0 \text{ and } \overline{s}(f) = 0 \Rightarrow (\overline{v} + \overline{s})f = \overline{v}(f) + \overline{s}(f) = 0$$

and

$$\overline{v}(f) = 0 \text{ and } g \in C^1(\Re^n) \Rightarrow (g\overline{v})f = 0.$$

We call the elements of this submodule the *killing fields* of f. A collection of basis elements for this submodule are called *infinitesimal generators*. A set of orthonormal vector fields is a "moving frame" in standard differential geometry terminology.

There are many ways to determine a basis set. Since the killing fields form a module in their own right, one can apply linear algebra to obtain

Theorem 5 *Suppose $f \in C^2(\Re^n)$. If $\nabla f(\overline{x})\,\nabla f(\overline{x})^T \neq 0 \quad \forall \overline{x} \in M_f$ then any $n-1$ columns (rows) of $I_{n\times n} - \nabla f^T(\nabla f\,\nabla f^T)^{-1}\nabla f$, where $I_{n\times n}$ is the matrix consisting of the constant function 1 on the diagonal and the zero function in all other entries, form a basis for the killing fields of f.*

Proof: See [1]. ∎

Example 6

Given

$$f(x,y,z) = ax + by + cz^2 \qquad a, c \neq 0.$$

Then $\overline{x} \in M_f$ if $z = \pm\sqrt{-\frac{ax+by}{c}}$. Choosing the positive root (the results are the same), and making this substitution, $\overline{v}(f) = 0$ gives

$$\overline{v}(f) = (a)v_x + (b)v_y + (2c\sqrt{-\frac{ax+by}{c}})v_z = 0.$$

Solving for v_x and eliminating this coefficient yields

$$\begin{aligned}\overline{v} &= v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\\ &= v_y\left(\frac{-b}{a}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) + v_z\left(\frac{2c\sqrt{\frac{ax+by}{-c}}}{-a}\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\right).\end{aligned}$$

The two vector fields on $\Re^3$ given by

$$\begin{aligned}\overline{\eta}_1 &= (\frac{-b}{a})\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\\ \overline{\eta}_2 &= (\frac{-2c\sqrt{-\frac{ax+by}{c}}}{a})\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\end{aligned}$$

form a basis for the killing fields of f.

Since the infinitesimal generators form a basis for the killing fields of f, every vector field $\overline{v}$ such that $\overline{v}(f) = 0$, with infinitesimal generators $\{\overline{\eta}_1, \ldots, \overline{\eta}_{n-1}\}$, can be written uniquely as

$$\overline{v} = \sum_{i=1}^{n-1} g_i\overline{\eta}_i \tag{5}$$

for some $g_i \in C^1(\Re^n)$ for $i = 1, \ldots, n-1$

3.4. Computation of Groups of Transformations from the Infinitesimal Generators

Let

$$\begin{aligned}{}_{\bullet}\varphi(\overline{x}) &: I \to \Re^n\\ &: \varepsilon \mapsto ({}_{\varepsilon}\varphi_{x_1}(\overline{x}), \ldots, {}_{\varepsilon}\varphi_{x_n}(\overline{x}))\end{aligned}$$

be a curve in $\Re^n$ satisfying ${}_0\varphi(\overline{x}) = \overline{x} \in M_f$. Recall ${}_{\varepsilon}\varphi$ is a curve in M_f if and only if

$$f({}_{\varepsilon}\varphi) = 0 \qquad \forall\varepsilon \in I.$$

Theorem 7 *Let ${}_0\varphi(\overline{x}) = \overline{x} \in M_f$. Then ${}_{\varepsilon}\varphi$ is a curve in M_f if and only if*

$$\frac{d_{\varepsilon}\varphi(\overline{x})}{d\varepsilon}\cdot\nabla f({}_{\varepsilon}\varphi(\overline{x})) = 0.$$

Proof: See [1]. ∎

Written in terms of components this condition reads

$$\begin{aligned}\frac{d_{\varepsilon}\varphi(\overline{x})}{d\varepsilon}\cdot\nabla f({}_{\varepsilon}\varphi(\overline{x})) &= \sum_{j=1}^{n}\frac{d_{\varepsilon}\varphi_{x_j}(\overline{x})}{d\varepsilon}\frac{\partial f}{\partial x_j}\\ &= \left[\sum_{j=1}^{n}\frac{d_{\varepsilon}\varphi_{x_j}(\overline{x})}{d\varepsilon}\frac{\partial}{\partial x_j}\right] f\\ &= 0.\end{aligned}$$

Letting

$$v_j({}_{\varepsilon}\varphi(\overline{x})) = \frac{d_{\varepsilon}\varphi_{x_j}(\overline{x})}{d\varepsilon}$$

results in

$$\left[\sum_{j=1}^{n}\frac{d_{\varepsilon}\varphi_{x_j}(\overline{x})}{d\varepsilon}\frac{\partial}{\partial x_j}\right] f = \left[\sum_{j=1}^{n} v_j({}_{\varepsilon}\varphi(\overline{x}))\frac{\partial}{\partial x_j}\right] f.$$

Using the identification

$$\overline{v} = (v_1, \ldots, v_n) \leftrightarrow \sum_{j=1}^{n} v_j\frac{\partial}{\partial x_j}$$

gives

Theorem 8 *If ${}_{\varepsilon}\varphi(\overline{x})$ is a curve in M_f and $\overline{v}$ is a vector field satisfying $\frac{d_{\varepsilon}\varphi_{x_j}(\overline{x})}{d\varepsilon} = v_j({}_{\varepsilon}\varphi(\overline{x}))$ for $j = 1, \ldots, n$ then $\overline{v}(f) = 0$. Conversely, if $\frac{d_{\varepsilon}\varphi_{x_j}(\overline{x})}{d\varepsilon} = v_j({}_{\varepsilon}\varphi(\overline{x}))$ for $j = 1, \ldots, n$, ${}_{\varepsilon}\varphi(\overline{x}) = \overline{x} \in M_f$ and $\overline{v}(f) = 0$ then ${}_{\varepsilon}\varphi(\overline{x})$ is a curve in M_f.*

This motivates

Definition 9 *The curve ${}_{\varepsilon}\varphi(\overline{x})$ in M_f is an integral curve of $\overline{v}$ provided that $\forall\varepsilon \in I$*

$$\frac{d_{\varepsilon}\varphi(\overline{x})}{d\varepsilon} = \overline{v}({}_{\varepsilon}\varphi(\overline{x})).$$

If ${}_0\varphi(\overline{x}) = ({}_0\varphi_{x_1}(\overline{x}), \ldots, {}_0\varphi_{x_n}(\overline{x})) = (x_1, \ldots, x_n) = \overline{x}$ one says the curve ${}_\varepsilon\varphi(\overline{x})$ starts at $\overline{x}$.

Using the previous theorem and the new terminology gives

Theorem 10 *The curve ${}_\varepsilon\varphi(\overline{x})$ in M_f is an integral curve of $\overline{v}$ starting at $\overline{x}$ if and only if for $j = 1, \ldots, n$*

$$\frac{d_\varepsilon\varphi_{x_j}(\overline{x})}{d\varepsilon} = v_j({}_\varepsilon\varphi_{x_1}(\overline{x}), \ldots, {}_\varepsilon\varphi_{x_n}(\overline{x})) \qquad {}_0\varphi_{x_j}(\overline{x}) = x_j. \tag{6}$$

Groups of transformations in M_f determined by a vector field $\overline{v}$ can now be found as follows: Let $\{\overline{\eta}_i\}_{i=1}^{n-1}$ form a basis for $\overline{v}$. Each $\overline{\eta}_i$ is (can be identified with) a vector field itself.

Solving the system of differential equations

$$\frac{d_\varepsilon\varphi(\overline{x})}{d\varepsilon} = \overline{\eta}_i({}_\varepsilon\varphi(\overline{x})) \qquad {}_0\varphi(\overline{x}) = \overline{x}$$

for ${}_\varepsilon\varphi(\overline{x})$ determines a group of transformations. If $g_i \in C^1(\Re^n)$ then $g_i\overline{\eta}_i$ is a vector field such that $g_i\overline{\eta}_i(f) = 0$ and the solution to

$$\frac{d_\varepsilon\varphi(\overline{x})}{d\varepsilon} = g_i\overline{\eta}_i({}_\varepsilon\varphi(\overline{x})) \qquad {}_0\varphi(\overline{x}) = \overline{x}$$

determines a curve in M_f, and hence a group of transformations of M_f. More generally, since the infinitesimal generators $\{\overline{\eta}_1, \ldots, \overline{\eta}_{n-1}\}$ form a basis for the vector fields $\overline{v}$ satisfying $\overline{v}(f) = 0$, then for any collection of functions

$$g_i \in C^1(\Re^n) \qquad i = 1, \ldots, n-1$$

it follows

$$\left(\sum_{i=1}^{n-1} g_i\overline{\eta}_i\right) f = 0$$

so the solution to the system of differential equations

$$\frac{d_\varepsilon\varphi(\overline{x})}{d\varepsilon} = \left(\sum_{i=1}^{n-1} g_i\overline{\eta}_i\right)({}_\varepsilon\varphi) \qquad {}_0\varphi(\overline{x}) = \overline{x}$$

determines a curve in M_f, and hence a group of transformations of M_f.

The process of solving the equations to find a group of transformations determined by the vector field $\overline{v}$ is called *the process of exponentiation*.

Example 11

For the problem

$$f(x, y, z) = ax + by + cz^2 = 0$$

it was previously determined one infinitesimal generator is

$$\overline{\eta}_2 = \left(\frac{-2c\sqrt{-\frac{ax+by}{c}}}{a}\right)\frac{\partial}{\partial x} + \frac{\partial}{\partial z}.$$

To obtain the group of transformations associated with this vector field solve

$$\begin{array}{rclcrcl} \frac{d_\varepsilon\varphi_x(\overline{x})}{d\varepsilon} & = & (\frac{-2c}{a})\sqrt{-\frac{a_\varepsilon\varphi_x + b_\varepsilon\varphi_y}{c}} & & {}_0\varphi_x & = & x \\ \frac{d_\varepsilon\varphi_y(\overline{x})}{d\varepsilon} & = & 0 & & {}_0\varphi_y & = & y \\ \frac{d_\varepsilon\varphi_z(\overline{x})}{d\varepsilon} & = & 1 & & {}_0\varphi_z & = & z. \end{array}$$

Solving the third equation gives

$${}_\varepsilon\varphi_z(x, y, z) = z + \varepsilon.$$

Substituting this into the first equation and integrating gives

$${}_\varepsilon\varphi_x(x, y, z) = x + (\frac{-2c}{a})(\sqrt{-\frac{ax+by}{c}}\varepsilon + \frac{\varepsilon^2}{2}).$$

The second equation obviously gives the identity map. Thus

$$\begin{aligned} {}_\varepsilon\varphi(\overline{x}) &= ({}_\varepsilon\varphi_x(\overline{x}), {}_\varepsilon\varphi_y(\overline{x}), {}_\varepsilon\varphi_z(\overline{x})) \\ &= (x + (\frac{-2c}{a})(\sqrt{-\frac{ax+by}{c}}\varepsilon + \frac{\varepsilon^2}{2}), y, z + \varepsilon). \end{aligned}$$

It can easily be verified by substitution that these maps define a group of transformations on M_f: Let $(x, y, z) \in M_f$ and

$$\begin{aligned} \hat{x} &\equiv {}_\varepsilon\varphi_x(x, y, z) = x + (\frac{-2c}{a})(\sqrt{-\frac{ax+by}{c}}\varepsilon + \frac{\varepsilon^2}{2}) \\ \hat{y} &\equiv {}_\varepsilon\varphi_y(x, y, z) = y \\ \hat{z} &\equiv {}_\varepsilon\varphi_z(x, y, z) = z + \varepsilon. \end{aligned}$$

Upon solving these equations for (x, y, z) and substituting into the original equation one obtains

$$a\hat{x} + b\hat{y} + c\hat{z}^2 = 0.$$

Hence $(\hat{x}, \hat{y}, \hat{z}) \in M_f$.

These results can be summarized by

Corollary 12 *Let $\overline{v}$ be a vector field satisfying $\overline{v}(f) = 0$. Each infinitesimal generator of $\overline{v}$ determines a curve in M_f.*

and

Corollary 13 *Let $\{{}_\varepsilon\varphi\}_{\varepsilon\in\Re}$ be a group of transformations of M_f determined by the process of exponentiating. If $f(\overline{x}) = 0$ then $f({}_\varepsilon\varphi(\overline{x})) = 0$.*

The conclusion of Corollary 13 is really just a tautology since a group of transformations of M_f means if $\overline{x} \in M_f$ then ${}_\varepsilon\varphi(\overline{x}) \in M_f$.

The set of all such transformations determined by equation (7) is the group of symmetries of M_f, denoted by S_{M_f}. Clearly it is the smallest group containing all of the groups "generated" by the infinitesimal generators $\{\overline{\eta}_1, \ldots, \overline{\eta}_{n-1}\}$ as subgroups. Furthermore, any transformation of M_f can be determined by solving such a system of equations.

Example 14

The family of curves generated by the vector field $\overline{v}(f)$ where $f(x, y, z) = ax + by + cz^2$, contains (among others) the family of curves $\{{}_\varepsilon\varphi(\overline{x})\}_{\overline{x}\in M_f}$ determined by $\overline{\eta}_2$

$$\begin{aligned} {}_\varepsilon\varphi(\overline{x}) &= \left({}_\varepsilon\varphi_x(\overline{x}), {}_\varepsilon\varphi_y(\overline{x}), {}_\varepsilon\varphi_z(\overline{x})\right) \\ &= \left(x - \frac{2c}{a}(\sqrt{\frac{ax+by}{-c}}\varepsilon + \frac{\varepsilon^2}{2}), y, z + \varepsilon\right). \end{aligned}$$

and the family of curves $\{_\varepsilon\varphi(\overline{x})\}_{\overline{x}\in M_f}$ determined by $\overline{\eta}_1$

$$\begin{aligned} {}_\varepsilon\varphi(\overline{x}) &= \left({}_\varepsilon\varphi_x(\overline{x}), {}_\varepsilon\varphi_y(\overline{x}), {}_\varepsilon\varphi_z(\overline{x})\right) \\ &= \left(x - (\frac{b}{a})\varepsilon, y+\varepsilon, z\right). \end{aligned}$$

However these are not the only curves. A simple transformation of the above set of curves produces a new family of curves $\{_\varepsilon\varphi(\overline{x})\}_{\overline{x}\in M_f}$ where

$$\begin{aligned} {}_\varepsilon\tilde{\varphi}(\overline{x}) &= \left({}_\varepsilon\tilde{\varphi}_x(\overline{x}), {}_\varepsilon\tilde{\varphi}_y(\overline{x}), {}_\varepsilon\tilde{\varphi}_z(\overline{x})\right) \\ &= \left(x, y - \frac{a}{b}\frac{2c}{a}(\sqrt{\frac{ax+by}{-c}}\varepsilon + \frac{\varepsilon^2}{2}), z+\varepsilon\right). \end{aligned}$$

This leads to the next topic.

3.5. The Group of Symmetries, S_{M_f}

It was observed that for an infinitesimal generator $\overline{\eta}_i$ of a vector field $\overline{v}$ satisfying $\overline{v}(f) = 0$, the solution to

$$\frac{d_\varepsilon\varphi(\overline{x})}{d\varepsilon} = \overline{\eta}_i({}_\varepsilon\varphi(\overline{x})) \qquad {}_0\varphi(\overline{x}) = \overline{x}$$

determines a group of transformations. If $g_i \in C^1(\Re^n)$ then $g_i\overline{\eta}_i$ is a vector field such that $g_i\overline{\eta}_i(f) = 0$ and the solution to

$$\frac{d_\varepsilon\varphi(\overline{x})}{d\varepsilon} = g_i\overline{\eta}_i({}_\varepsilon\varphi(\overline{x})) \qquad {}_0\varphi(\overline{x}) = \overline{x}$$

determines a curve in M_f, and hence a group of transformations of M_f. More generally, since the infinitesimal generators $\{\overline{\eta}_1, \ldots, \overline{\eta}_{n-1}\}$ form a basis for the vector fields $\overline{v}$ satisfying $\overline{v}(f) = 0$, then for any collection of functions

$$g_i \in C^1(\Re^n) \qquad i = 1, \ldots, n-1$$

it follows

$$\left(\sum_{i=1}^{n-1} g_i\overline{\eta}_i\right) f = 0$$

so the solution to the system of differential equations

$$\frac{d_\varepsilon\varphi(\overline{x})}{d\varepsilon} = \left(\sum_{i=1}^{n-1} g_i\overline{\eta}_i\right)({}_\varepsilon\varphi) \qquad {}_0\varphi(\overline{x}) = \overline{x}$$

determines a curve in M_f, and hence a group of transformations of M_f.

The set of all such transformations determined by this equation is the group of symmetries of M_f, denoted by S_{M_f}. Clearly it is the smallest group containing all of the groups "generated" by the infinitesimal generators $\{\overline{\eta}_1, \ldots, \overline{\eta}_{n-1}\}$ as subgroups. Furthermore, any transformation of M_f can be determined by solving such a system of equations.

3.6. Invariant Functions and their Calculation

Suppose one is given the equation $f = 0$. Let

$$\begin{aligned} \Gamma(\bullet) &: S_{M_f} \times M_f \to M_f \\ &: ({}_\varepsilon\varphi, \overline{x}) \mapsto {}_\varepsilon\varphi(\overline{x}) \end{aligned}$$

be the S_{M_f}–action on M_f. Then S_{M_f} acts on $\hom(M_f, \Re)$ in a natural way

$$\begin{aligned} \hat{\Gamma}(\bullet) &: S_{M_f} \times \hom(M_f, \Re) \to \hom(M_f, \Re) \\ &: ({}_\varepsilon\varphi, \Phi) \mapsto {}_\varepsilon\varphi * \Phi \end{aligned}$$

where

$$\begin{aligned} [{}_\varepsilon\varphi * \Phi](\bullet) &: M_f \to \Re \\ &: \overline{x} \mapsto \Phi({}_\varepsilon\varphi(\overline{x})). \end{aligned}$$

A simple calculation shows $\hat{\Gamma}$ is a group action.

Definition 15 *An element $\Phi \in \hom(M_f, \Re)$ is an S_{M_f}–invariant of $\hom(M_f, \Re)$ if Φ is invariant under the action of S_{M_f} on $\hom(M_f, \Re)$. In other words, the stabilizer of Φ is S_{M_f}*

$$\{{}_\varepsilon\varphi \in S_{M_f} : {}_\varepsilon\varphi * \Phi = \Phi\} = S_{M_f}.$$

From this definition and the given induced action it follows

Theorem 16 *Let S_{M_f} be a group acting on a set $\hom(M_f, \Re)$. An element $\Phi \in \hom(M_f, \Re)$ is an absolute S_{M_f}–invariant of $\hom(M_f, \Re)$ if and only if*

$$\Phi({}_\varepsilon\varphi(\overline{x})) = \Phi(\overline{x}) \qquad \forall {}_\varepsilon\varphi \in S_{M_f}.$$

This necessary and sufficient condition is often taken as the definition of an absolute invariant function. Though the definition of an invariant element of the set $\hom(M_f, \Re)$ should be expressed in terms of the more fundamental action

$$\begin{aligned} \Gamma(\bullet) &: S_{M_f} \times M_f \to M_f \\ &: ({}_\varepsilon\varphi, \overline{x}) \mapsto {}_\varepsilon\varphi(\overline{x}). \end{aligned}$$

The following theorem gives a necessary and sufficient condition for such an absolute invariant function.

Theorem 17 *Let $\overline{\eta}_i$ for $i = 1, \ldots, n-1$ be the infinitesimal generators for the killing fields of f. Then $\Phi \in \hom(M_f, \Re)$ is an absolute S_{M_f}-invariant function if and only if $\overline{\eta}_i(\Phi) = 0$ for $i = 1, \ldots, n-1$.*

Proof: See [1]. ■

Computationally invariant functions can be calculated by integrating the *characteristic equation* of each infinitesimal generator. Given the generator $\overline{\overline{\eta}} \equiv \overline{\eta}_i(\overline{x})\frac{\partial}{\partial x_i} + \overline{\eta}_j(\overline{x})\frac{\partial}{\partial x_j}$ the characteristic equation is

$$\frac{dx_i}{\overline{\eta}_i(\overline{x})} = \frac{dx_j}{\overline{\eta}_j(\overline{x})}. \qquad (7)$$

Example 18

For the standard example problem, the two infinitesimal generators are

$$\begin{aligned} \overline{\eta}_1 &\equiv (\frac{-b}{a})\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \\ \overline{\eta}_2 &\equiv (\frac{-2c\sqrt{-\frac{ax+by}{c}}}{a})\frac{\partial}{\partial x} + \frac{\partial}{\partial z}. \end{aligned}$$

For the first tangent vector, $\overline{\eta}_1$, the characteristic equation is

$$\frac{dx}{(\frac{-b}{a})} = \frac{dy}{1}.$$

Integrating gives

$$ax + by = c_1$$

where c_1 is the constant of integration. Since Φ is an invariant function, it must be constant under the given transformation, hence an invariant function under the group of transformations

$$x \mapsto x - \frac{b}{a}\varepsilon, \quad y \mapsto y + \varepsilon, \quad z \mapsto z$$

must be a function with parameters of the form $ax + by$, that is

$$g_1(ax + by, z)$$

where g_1 is an arbitrary function.

For the second tangent vector, $\overline{\eta}_2$, the characteristic equation is

$$\frac{dx}{(\frac{-2c}{a}\sqrt{-\frac{ax+by}{c}})} = \frac{dz}{1}.$$

Integrating shows an invariant function under the transformation group determined by this generator must be of the form

$$g_2(ax + by + cz^2)$$

where g_2 is an arbitrary function.

An invariant function *under both groups* of transformations must be of the form

$$\Phi = g_3(ax + by + cz^2)$$

where g_3 is an arbitrary function. Thus there are only trivial invariants defined on the set of all roots of the function

$$f(x, y, z) = ax + by + cz^2.$$

3.7. Lie group analysis of Differential Equations

These concepts and theorems can be extended to address the problem of finding the group of symmetries and invariant functions of differential equations. These tend to be more difficult to solve because to determine the coefficients of the vector field $\overline{v}$ one must solve a system of partial differential equations.

Lie group analysis determines the infinitesimal generators of the (connected) symmetry group of the conservation equation. Once these generators are known, the invariants Φ can be determined by the necessary and sufficient condition

$$\overline{\eta}_i(\Phi) = 0 \qquad (8)$$

for each infinitesimal generator $\overline{\eta}_i$ [6, 7]. The infinitesimal generators form a basis for the tangent vector $\overline{v}$ described below. (The “infinitesimal generators” are called generators because they generate the transformation groups associated with the symmetries of the differential equation.) The procedure will determine all 1-parameter continuous transformation groups of the differential equation. Discrete symmetries such as reflection will not be found. Given $\overline{s}$ is the set of all dependent variables, $\overline{x}$ is the set of all (dependent and independent) variables, and $|\overline{x}|$ is the number of variables in $\overline{x}$, then the computational procedure is as follows:

Form the tangent vector

$$\overline{v} = \sum_{j=1}^{|\overline{x}|} v_{x_j}(\overline{x})\frac{\partial}{\partial x_j}. \qquad (9)$$

If the differential equation is of order n, then calculate the n^{th} prolongation of the tangent vector,

$$Pr^{(n)}\overline{v} = \overline{v} + \sum_{i=1}^{|\overline{s}|}\sum_{\overline{j}\in\overline{\overline{J}}_{s_i}} \psi_{s_i}^{\overline{j}}(\overline{x})\frac{\partial}{\partial s_i^{(\overline{j})}} \qquad (10)$$

$\forall\,\overline{j} = (j_1, \ldots, j_k)$ such that $0 \le j_k \le |s_i|$, and $1 \le k \le n$, and $\overline{j} \neq (0, \ldots, 0)$. $\overline{\overline{J}}_{s_i}$ is the set of sets of all unordered multi-indices corresponding to the partials with respect to the parameters of the dependent variables $(s_i^{(\overline{m})}, \quad m = 1, \ldots, |s_i|)$, and where

$$\begin{aligned}\psi_h^{\overline{j}}(\overline{x}) \quad &= \quad D_{\overline{j}}\left[v_h(\overline{x}) - \sum_{i=1}^{|h|} v_{h_i}(\overline{x})\frac{\partial h}{\partial h_i}\right] \\ &\quad + \sum_{i=1}^{|h|} v_{h_i}(\overline{x})\frac{\partial^{k+1}h}{\partial h_i h_{j_1} h_{j_2}\ldots h_{j_k}}.\end{aligned}$$

The coefficients, v_{x_j}, are determined by the requirement

$$(Pr^{(n)}\overline{v})(f) = 0. \qquad (11)$$

This more complicated form reduces to the tangent vector in the case of non-differential equations. This technique has been applied to the complete differential form of the conservation statement.

4. Lie Group Analysis of the Conservation Equation

Section 3 provides the necessary theory and concepts for performing Lie group analysis of a differential equation such as the conservation equation (eq. (1)). The finite-difference approximation of the conservation equation was analyzed in [1]. The associated transformations and absolute invariants were found. The following section details the current state of the Lie group analysis of the full differential form of the conservation equation.

Starting from equation (1), and relabeling

$$\begin{array}{lcllcl} W_s & \equiv & W_s\cos\theta & W_l & \equiv & W_l A_{\text{sky}} \\ c_1 & \equiv & \alpha_s & c_2 & \equiv & \alpha_l \\ c_3 & \equiv & -\epsilon\sigma & c_4 & \equiv & k \\ c_5 & \equiv & -C_T & & & \end{array}$$

the conservation equation is

$$f = c_1 W_s + c_2 W_l + c_3 T_s^4 + h(T_\infty - T_s) + c_4\frac{\partial T_s}{\partial z} + c_5\frac{\partial T_s}{\partial t}. \qquad (12)$$

The conservation equation, f, is a first order partial

f	The conservation of energy equation
$\overline{r}$	The set of independent variables
$\overline{s}$	The set of dependent variables
$\overline{x}$	The set of variables (dependent and independent)
x_j	Individual variables from $\overline{x}$
$\lvert\overline{x}\rvert$	Length of the vector, $\overline{x}$.
v_{x_j}	The prolongation variables associated with each x_j
$v_{x_j}^{\{t,z\}}$	The partial of v_{x_j} with respect to t and z
$D_z(f)$	The total derivative of f with respect to z

Table 1. Variables, definitions, and notations used in this section.

differential equation (PDE) with seven variables, $\overline{x} = \{W_s, W_l, h, T_\infty, T_s, t, z\}$. The independent variables are $\overline{r} = \{W_s, W_l, h, T_\infty, t, z\}$, and the dependent variables are $\overline{s} = \{T_s\}$.

The tangent vector (operator), $\overline{v}$, is found by using equation (9) from Section 3.7,

$$\begin{aligned}\overline{v} &= v_{W_s}\frac{\partial}{\partial W_s} + v_{W_l}\frac{\partial}{\partial W_l} + v_h\frac{\partial}{\partial h}\\ &\quad + v_{T_\infty}\frac{\partial}{\partial T_\infty} + v_{T_s}\frac{\partial}{\partial T_s} + v_t\frac{\partial}{\partial t} + v_z\frac{\partial}{\partial z}\end{aligned}$$

where the $v_{x_j}(\overline{x})$ are unknown coefficients (arbitrary functions of all the variables) that we seek to determine such that the infinitesimal criterion (the first prolongation for a first order PDE) is satisfied $\forall\overline{x}$ lying on the manifold determined by the conservation equation. The infinitesimal criterion may be found by applying the first prolongation

$$\begin{aligned}Pr^{(1)}\overline{v} &= v_{W_s}\frac{\partial}{\partial W_s} + v_{W_l}\frac{\partial}{\partial W_l} + v_h\frac{\partial}{\partial h} + v_{T_\infty}\frac{\partial}{\partial T_\infty}\\ &\quad + v_{T_s}\frac{\partial}{\partial T_s} + v_t\frac{\partial}{\partial t} + v_z\frac{\partial}{\partial z} \qquad (13)\\ &\quad + \left(v_{T_s}^{\{z\}} - T_s^{\{t\}}\left(v_t^{\{z\}} + T_s^{\{z\}} v_t^{\{T_s\}}\right)\right.\\ &\qquad \left. - T_s^{\{z\}}\left(v_z^{\{z\}} + T_s^{\{z\}} v_z^{\{T_s\}}\right) + T_s^{\{z\}} v_{T_s}^{\{T_s\}}\right)\frac{\partial}{\partial T_s^{\{z\}}}\\ &\quad + \left(v_{T_s}^{\{t\}} - T_s^{\{t\}}\left(v_t^{\{t\}} + T_s^{\{t\}} v_t^{\{T_s\}}\right)\right.\\ &\qquad \left. - T_s^{\{z\}}\left(v_z^{\{t\}} + T_s^{\{t\}} v_z^{\{T_s\}}\right) + T_s^{\{t\}} v_{T_s}^{\{T_s\}}\right)\frac{\partial}{\partial T_s^{\{t\}}}\end{aligned}$$

to the conservation equation,

$$\begin{aligned}Pr^{(1)}\overline{v}(f) &= v_{W_s}c_1 + v_{W_l}c_2 + v_h(T_\infty - T_s) + v_{T_\infty}h\\ &\quad + v_{T_s}(4c_3T_s^3 - h) + v_t(0) + v_z(0)\\ &\quad + \left(v_{T_s}^{\{z\}} - T_s^{\{t\}}\left(v_t^{\{z\}} + T_s^{\{z\}} v_t^{\{T_s\}}\right)\right.\\ &\qquad \left. - T_s^{\{z\}}\left(v_z^{\{z\}} + T_s^{\{z\}} v_z^{\{T_s\}}\right) + T_s^{\{z\}} v_{T_s}^{\{T_s\}}\right)c_4\\ &\quad + \left(v_{T_s}^{\{t\}} - T_s^{\{t\}}\left(v_t^{\{t\}} + T_s^{\{t\}} v_t^{\{T_s\}}\right)\right.\\ &\qquad \left. - T_s^{\{z\}}\left(v_z^{\{t\}} + T_s^{\{t\}} v_z^{\{T_s\}}\right) + T_s^{\{t\}} v_{T_s}^{\{T_s\}}\right)c_5\\ &= 0. \qquad (14)\end{aligned}$$

The prolongation can be restricted to the manifold by first obtaining an expression for one of the variables (which has a non-vanishing coefficient) in terms of the other variables, as determined by the conservation equation. Generally, the least complicated derivations will be necessary if one solves for the partial derivative, say $T_s^{\{t\}}$. Thus, from the conservation equation,

$$T_s^{\{t\}} = -\frac{c_1W_s + c_2W_l + c_3T_s^4 + h(T_\infty - T_s) + c_4T_s^{\{z\}}}{c_5}. \qquad (15)$$

Substituting this expression into the infinitesimal criterion of equation (14) gives

$$\begin{aligned}Pr^{(1)}\overline{v}(f) &= v_{W_s}c_1 + v_{W_l}c_2 + v_h(T_\infty - T_s) + v_{T_\infty}h\\ &\quad + v_{T_s}(4c_3T_s^3 - h) + c_5\left(v_{T_s}^{\{t\}} - T_s^{\{z\}} v_z^{\{t\}}\right)\\ &\quad + c_4\left(v_{T_s}^{\{z\}} - T_s^{\{z\}}\left(v_z^{\{z\}} + T_s^{\{z\}} v_z^{\{T_s\}}\right) + T_s^{\{z\}} v_{T_s}^{\{T_s\}}\right)\\ &\quad - c_5(T_s^{\{t\}})^2 v_t^{\{T_s\}} - T_s^{\{t\}}\left(c_4\left(v_t^{\{z\}} + T_s^{\{z\}} v_t^{\{T_s\}}\right)\right.\\ &\quad \left. + c_5\left(v_t^{\{t\}} + T_s^{\{z\}} v_z^{\{T_s\}} - v_{T_s}^{\{T_s\}}\right)\right)\\ &= 0.\end{aligned}$$

This equation must hold for all $T_s^{\{z\}}$. Therefore, the functions corresponding to the coefficients of $T_s^{\{z\}}$ must be zero. Consequently, after collecting this monomial and the constant monomial, it is apparent that the following conditions must be satisfied

$$\begin{aligned}\frac{c_4}{c_5}\left(c_4\, v_t^{\{z\}} - c_5\, v_z{}^{\{z\}}\right) + \left(c_4\, v_t{}^{\{t\}} - c_5\, v_z{}^{\{t\}}\right) & \qquad (16)\\ - \frac{r_c}{c_5}\left(c_4\, v_t^{\{T_s\}} - c_5\, v_z{}^{\{T_s\}}\right) &= 0\end{aligned}$$

$$\begin{aligned}c_1\, v_{W_s} + c_2\, v_{W_l} + h(v_{T_\infty} - v_{T_s}) + v_h\,(T_\infty - T_s) & \\ + 4\, c_3\, T_s{}^3\, v_{T_s} + \frac{c_4}{c_5}\left(r_c\, v_t^{\{z\}} - c_5\, v_{T_s}{}^{\{z\}}\right) & \qquad (17)\\ + \left(r_c\, v_t^{\{t\}} - c_5\, v_{T_s}{}^{\{t\}}\right) - \frac{r_c}{c_5}\left(r_c\, v_t^{\{T_s\}} - c_5\, v_{T_s}{}^{\{T_s\}}\right) &= 0\end{aligned}$$

where

$$r_c \equiv c_1\, W_s + c_2\, W_l + h\,(T_\infty - T_s) + c_3\, T_s{}^4.$$

A basis ($\overline{\overline{\eta}}$) can be found after finding the most general solution for the unknowns, $v_{x_j}(\overline{x})$. The most general solution for equation (16) is

$$v_z = \frac{c_4}{c_5}v_t + g_1$$

where g_1 is an arbitrary function satisfying

$$c_4\, g_1^{\{z\}} + c_5\, g_1^{\{t\}} - r_c\, g_1^{\{T_s\}} = 0.$$

The exact form of g_1 need not be found since v_z does not appear in the second equation. Equation (17) is a nonlinear PDE that cannot be solved analytically with the current state of the art. However, for very short time periods, one may approximate $W_s, W_l, h,$ and T_∞ as constant. If an invariant is found under these conditions, then resources can be allocated to attempt to generalize it. Conversely, if no invariants are found under these conditions, it is highly unlikely that they will exist in general because the approximation only involves variables that are independent of T_s. Using this approximation, the conservation equation may be rewritten as

$$c_1 U^{\{t\}} + c_2 U^{\{z\}} + c_3 U + c_4 U^4 + c_5 = 0 \qquad (18)$$

where U represents T_s. The tangent vector for this equation is

$$\overline{v} = v_U \frac{\partial}{\partial U} + v_t \frac{\partial}{\partial t} + v_z \frac{\partial}{\partial z}$$

and the infinitesimal criterion yields a system of equations are very similar to equation (16) and equation (17)

$$\begin{aligned} \frac{c_2}{c_1}(c_2 v_t^{\{z\}} - c_1 v_z{}^{\{z\}}) + (c_2 v_t{}^{\{t\}} - c_1 v_z{}^{\{t\}}) & \qquad (19) \\ - \frac{r_c}{c_1}(c_2 v_t^{\{U\}} - c_1 v_z{}^{\{U\}}) &= 0 \\ (c_3 + 4 c_4 U^3) v_U + \frac{c_2}{c_1}(r_c v_t^{\{z\}} - c_1 v_U{}^{\{z\}}) & \qquad (20) \\ +(r_c v_t^{\{t\}} - c_1 v_U{}^{\{t\}}) - \frac{r_c}{c_1}(r_c v_t^{\{U\}} - c_1 v_U{}^{\{U\}}) &= 0 \end{aligned}$$

where

$$r_c \equiv c_3 U + c_4 U^4 + c_5.$$

A basis ($\overline{\overline{\eta}}$) can be found after finding the most general solution for the unknowns, $v_{x_j}(\overline{x})$. The most general solution for equation (19) is

$$v_z = \frac{c_2}{c_1} v_t + g_1$$

where g_1 is an arbitrary function satisfying

$$c_1 g_1^{\{t\}} + c_2 g_1^{\{z\}} - r_c g_1^{\{U\}} = 0.$$

The exact form of g_1 need not be found since v_z does not appear in the second equation. The second equation may be solved for v_U,

$$v_U = r_c \left(-\frac{1}{c_1} v_t + g_2 \right)$$

where g_2 is an arbitrary function satisfying

$$c_1 g_2^{\{t\}} + c_2 g_2^{\{z\}} - r_c g_2^{\{U\}} = 0$$

although all that really need be known about g_2 is that it is an arbitrary function.

These solutions are substituted into the tangent vector,

$$\begin{aligned} \overline{v} &= v_U \frac{\partial}{\partial U} + v_t \frac{\partial}{\partial t} + v_z \frac{\partial}{\partial z} \\ &= \left(r_c \left(-\frac{1}{c_1} v_t + g_2 \right) \right) \frac{\partial}{\partial U} + v_t \frac{\partial}{\partial t} + \left(\frac{c_2}{c_1} v_t + g_1 \right) \frac{\partial}{\partial z} \\ &= g_1 \left(\frac{\partial}{\partial z} \right) + g_2 \left(r_c \frac{\partial}{\partial U} \right) + v_t \left(\frac{\partial}{\partial t} - \frac{r_c}{c_1} \frac{\partial}{\partial U} + \frac{c_2}{c_1} \frac{\partial}{\partial z} \right). \end{aligned}$$

Therefore, the basis is

$$\begin{aligned} \eta &= \left\{ \frac{\partial}{\partial z}, r_c \frac{\partial}{\partial U}, -\frac{r_c}{c_1} \frac{\partial}{\partial U} + \frac{\partial}{\partial t} + \frac{c_2}{c_1} \frac{\partial}{\partial z} \right\} \\ &= \left\{ \frac{\partial}{\partial t}, \frac{\partial}{\partial z}, r_c \frac{\partial}{\partial U} \right\}. \end{aligned}$$

A constant function and the original conservation equation are the only solutions of these three generators. Therefore, only trivial invariants exist for this differential form of the conservation equation. Similar results were found for the finite-difference form of the conservation equation.

5. Conclusions

Prior to this body of research, no published attempts have been made at applying Lie group analysis to LWIR imagery. The current work considers absolute intensity invariants of a single point in LWIR imagery and it is not surprising that only trivial invariants exist for this case. The techniques of Lie group analysis provides a powerful tool for constructing functions that can serve as classifier features for object recognition problems. An advantage of these techniques is that they are not sensor specific. Thus, this approach is applicable to data from all parts of the frequency spectrum.

This work, along with further developments, applications, and experiments in [1], form the foundation of the theory needed for future work in simultaneous covariants (a generalization of relative invariants and absolute invariants). The theory of simultaneous covariants will allow us to (analytically) consider the more general cases of several points changing together through time. Furthermore, the rigorous definition of quasi-invariants developed in [2] will provide empirical solutions that are equally useful. We believe that these are the tools and techniques which will allow the next generation of object recognition systems to handle general scenarios and sensors.

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